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The Hodge star operator on Schubert forms

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Abstract

Let $X = G/P$ be a homogeneous space of a complex semisimple Lie group G equipped with a hermitian metric. We study the action of the Hodge star operator on the space of harmonic differential forms on X. We obtain explicit combinatorial formulas for this action when X is an irreducible hermitian symmetric space of compact type. © 2002 Elsevier Science Ltd. All rights reserved.

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0. Introduction

Let us recall the definition of the Hodge $*$-operator. If V is an n-dimensional Euclidean vector space, choose an orthonormal basis $e_1, \dots, e_n$ of V and define the star operator $* : \bigwedge^k V \to \bigwedge^{n-k} V$ by

$$*(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}) = \operatorname{sgn}(\sigma) e_{\sigma(k+1)} \wedge \cdots \wedge e_{\sigma(n)}$$

for any permutation σ of the indices $(1, \dots, n)$. The operator $*$ depends only on the inner product structure of V and the orientation determined by the basis $e_1, \dots, e_n$. If V is the real vector space underlying a hermitian space then $*$ is defined using the natural choice of orientation coming from the complex structure.

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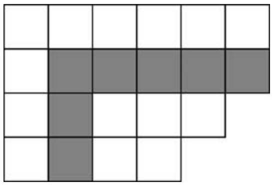

Fig. 1. A hook in the diagram of $(6,6,5,4)$.

Let X be a hermitian complex manifold of complex dimension d, and $A^{p,q}(X)$ the space of complex valued smooth differential forms of type (p,q) on X. The star operator taken pointwise gives a complex linear isomorphism $* : A^{p,q}(X) \to A^{d-q,d-p}(X)$ such that $** = (-1)^{p+q}$. Since $*$ commutes with the Laplacian Δ, it induces an isomorphism $\mathscr{H}^k(X) \to \mathscr{H}^{2d-k}(X)$ between spaces of harmonic forms on X. This in turn gives a map on cohomology groups $* : H^k(X,\mathbb{C}) \to H^{2d-k}(X,\mathbb{C})$ which depends on the metric structure of X.

Our main goal is to compute the action of $*$ explicitly when X is an irreducible hermitian symmetric space of compact type, equipped with an invariant Kähler metric. These spaces have been classified by É. Cartan; there are four infinite families and two 'exceptional' cases. We will describe our result here in the case of the Grassmannian

$$G(m,n) = U(m+n)/(U(m) \times U(n))$$

of complex m-dimensional linear subspaces of $\mathbb{C}^{m+n}$.

Recall that a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k)$ is identified with its Young diagram of boxes; the *weight* $|\lambda|$ is the number of boxes in λ. Given a diagram λ and a box $x \in \lambda$, the *hook* H_x is the set of all boxes directly to the right and below x, including x itself (see Fig. 1). The number of boxes in H_x is the *hook length* h_x. We let

$$h_\lambda := \prod_{x \in \lambda} h_x$$

denote the product of the hook lengths in λ. It is known that $N_\lambda = |\lambda|!/h_\lambda$ is the dimension of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to λ. The integer N_λ also counts the number of *standard Young tableaux* on λ, that is, the number of different ways to fill the boxes in λ with the numbers $1, 2, \ldots, |\lambda|$ so that the entries are strictly increasing along rows and columns. This fact is due to Frame et al. [7].

Partitions parametrize the harmonic forms corresponding to the Schubert classes, which are the natural geometric basis for the cohomology ring of $G = G(m,n)$. For each partition λ whose diagram is contained in the $m \times n$ rectangle (n^m), there is a harmonic form Ω_λ of type $(|\lambda|, |\lambda|)$ which is dual to the class of the codimension $|\lambda|$ Schubert variety X_λ in G. The Poincaré dual of Ω_λ (i.e. the dual form with respect to the Poincaré pairing $(\phi, \psi) \mapsto \int_G \phi \wedge \psi$) corresponds to the diagram λ' which, when inverted, is the complement of λ in (n^m) (see Fig. 2).

Normalize a given invariant hermitian metric on G so that its fundamental form is the Schubert form Ω_1. We can now state our result for the action of the star operator.

$$*\Omega_\lambda = \frac{h_\lambda}{h_{\lambda'}} \Omega_{\lambda'}. \tag{1}$$

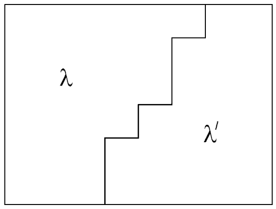

Fig. 2. The diagrams for Poincaré dual forms on $G(6,8)$.

There are similar results for infinite families of different type: the even orthogonal and Lagrangian Grassmannians. In these cases combinatorialists have identified the correct notions of 'hook' and 'hook length', and our formula for $*$ is a direct analogue of (1). We also compute the action of $*$ for quadric hypersurfaces and the exceptional cases.

More generally, our calculations are valid for any Kähler manifold whose cohomology ring coincides with that of a hermitian symmetric space. For example we compute the action of $*$ on the harmonic forms (with respect to any Kähler metric) for the odd orthogonal Grassmannians $SO(2n+1)/U(n)$.

The motivation for this work came from Arakelov geometry. A combinatorial understanding of the Lefschetz theory on homogeneous spaces is useful in the study of the corresponding objects over the ring of integers. For such arithmetic varieties, Gillet and Soulé have formulated analogues of Grothendieck's standard conjectures on algebraic cycles (see [21, Section 5.3]). Our calculation of $*$ has been used by Kresch and the second author [13] to verify these conjectures for the arithmetic Grassmannian $G(2,n)$.

Let us briefly outline the contents of the paper. In the first section we state our main theorem about the action of the Hodge star operator on Schubert forms on irreducible compact hermitian symmetric spaces. In the next section, we show that the star operator on these spaces maps a Schubert form to a non-zero multiple of the Poincaré dual form. Our proof is based on Kostant's construction of a basis of the space of harmonic forms which is recalled here. We prove our main theorem in section three. In the fourth section, we use our calculation of the star operator to give an explicit formula for the adjoint of the Lefschetz operator. Here we recover some results of Proctor. The example in the final section shows that for complete flag varieties, the star operator no longer maps a Schubert form to a multiple of the Poincaré dual form.

1. Statement of the main theorem

We begin with some more notation from combinatorics: a partition $\lambda = (\lambda_i)_{i \geqslant 1}$ is *strict* if its parts λ_i are distinct; the number of non-zero parts is the *length* of λ, denoted $\ell(\lambda)$. Define $\alpha(\lambda) = |\lambda| - \ell(\lambda)$ to be the number of boxes in λ that are not in the first column.

For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0)$, the *shifted diagram* $S(\lambda)$ is obtained from the Young diagram of λ by moving the ith row $(i-1)$ squares to the right, for each $i > 1$. The *double diagram* $D(\lambda)$ consists of $S(\lambda)$ dovetailed into its reflection in the main diagonal $\{(i,i) : i > 0\}$; in Frobenius notation, we have $D(\lambda) = (\lambda_1, \ldots, \lambda_m \,|\, \lambda_1 - 1, \ldots, \lambda_m - 1)$ (this is illustrated in Fig. 3; see also [15, Section I.1] for Frobenius notation). For each box x in $S(\lambda)$,

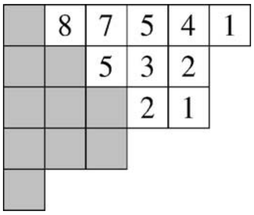

Fig. 3. $D(\lambda)$, $S(\lambda)$ and hook lengths for $\lambda=(5,3,2)$.

the hook length h_x is defined to be the hook length at x in the double diagram $D(\lambda)$. Fig. 3 displays these hook lengths for $\lambda=(5,3,2)$.

We let

$$g_\lambda := \prod_{x\in S(\lambda)} h_x$$

the product over all boxes x of the shifted diagram of λ. We remark that $|\lambda|!/g_\lambda$ counts the number of standard shifted tableaux of shape λ; it also occurs in the degree formula for the corresponding projective representation of the symmetric group. See for instance [10, p. 187 and Theorem 10.7] for definitions and details.

The irreducible compact hermitian symmetric spaces have been classified by Cartan [3]; there are four infinite families and two exceptional cases (see [24] for a modern treatment, in particular Corollary 8.11.5). We will recall this list; each such space is of the form $G/P=K/V$, with notation as in Section 2 (we will use the compact presentation K/V). In each case we have a natural K-invariant hermitian metric, unique up to positive scalar. The K-invariant differential forms coincide with the harmonic forms for this metric, and are all of (p,p) type for some p.

The Schubert cycles form a basis for the integral homology ring; their duals in cohomology are represented by unique harmonic forms, called *Schubert forms*. The Schubert forms are parametrized by the set W^1 defined in (4) below, and for the first three infinite families that follow this parameter space can be realized using integer partitions. We refer to [1,4,8], [18, Section 12] for more information.

(i) The Grassmannian $G(m,n)=U(m+n)/(U(m)\times U(n))$ of m-dimensional linear subspaces of $\mathbb{C}^{m+n}$, with $\dim_{\mathbb{C}} G(m,n)=mn$. We have a Schubert form Ω_λ of type $(|\lambda|,|\lambda|)$ for each partition λ whose Young diagram is contained in the $m\times n$ rectangle $(n^m)=(n,\dots,n)$. The Poincaré dual form corresponds to the diagram λ' described in the introduction.

(ii) The even orthogonal Grassmannian $OG(n,2n)=SO(2n)/U(n)$ (spinor variety) parametrizing maximal isotropic subspaces of $\mathbb{C}^{2n}$ equipped with a nondegenerate symmetric form, with $\dim_{\mathbb{C}} OG(n,2n)=\binom{n}{2}$. There is a Schubert form Φ_λ of type $(|\lambda|,|\lambda|)$ for each strict partition λ whose diagram is contained in the triangular partition $\rho_{n-1}:=(n-1,n-2,\dots,1)$. The Poincaré dual of Φ_λ corresponds to the strict partition λ' whose parts complement the parts of λ in the set $\{1,\dots,n-1\}$.

(iii) The Lagrangian Grassmannian $LG(n,2n)=Sp(2n)/U(n)$ parametrizing Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with a symplectic form, with $\dim_{\mathbb{C}} LG(n,2n)=\binom{n+1}{2}$. Here we have a Schubert form of type $(|\lambda|,|\lambda|)$, denoted Ψ_λ, for each strict partition λ whose diagram is contained in $\rho_n = (n,\ldots,1)$. The Poincaré dual of Ψ_λ corresponds to the strict partition λ' whose parts complement the parts of λ in the set $\{1,\ldots,n\}$.

(iv) The complex quadric $Q(n)=SO(n+2)/(SO(n)\times SO(2))$, isomorphic to a smooth quadric hypersurface in $\mathbb{P}^{n+1}(\mathbb{C})$, of dimension n. Let ω denote the Kähler form which is dual to the class of a hyperplane. Our reference for the cohomology ring of $Q(n)$ is [5, Section 2] (working in the context of Chow rings). There are two cases depending on the parity of n:

- If $n=2k-1$ is odd then there is one Schubert form $e \in \mathscr{H}^{2k}(X)$ and $\omega^k=2e$. The complete list of Schubert forms is

  $$1,\omega,\omega^2,\ldots,\omega^{k-1},e,\omega e,\ldots,\omega^{k-1}e.$$

  The Poincaré dual of ω^i is $\omega^{k-1-i}e$, for $0 \leqslant i \leqslant k-1$.
- If $n=2k$ is even then there are two distinct Schubert forms $e_0, e_1 \in \mathscr{H}^{2k}(X)$ which correspond to the two rulings of the quadric (in homology). Furthermore $\omega^k = e_0 + e_1$ and the complete list of Schubert forms is

  $$1,\omega,\omega^2,\ldots,\omega^{k-1},e_0,e_1,\omega e_0=\omega e_1,\omega^2 e_0,\ldots,\omega^k e_0.$$

  The Poincaré dual of ω^i is $\omega^{k-i}e_0$, for $0 \leqslant i \leqslant k-1$, while the dual of e_j is e_{j+k}, where the indices are taken mod 2.

(v) The 'exceptional' space $E_6/(SO(10)\cdot SO(2))$, of complex dimension 16.

(vi) The 'exceptional' space $E_7/(E_6\cdot SO(2))$, of complex dimension 27.

Suppose X is a compact Kähler manifold whose cohomology ring is isomorphic to any occurring in the preceding examples. We will see in Section 4 that the action of the star operator on $H^*(X,\mathbb{C})$ is the same as if X were a hermitian symmetric space. If we look among the homogeneous spaces G/P considered in Section 2 we find one example with this property which is not itself hermitian symmetric:

(ii$'$) The odd orthogonal Grassmannian $OG(n-1,2n-1)=SO(2n-1)/U(n-1)$ parametrizing maximal isotropic subspaces of $\mathbb{C}^{2n-1}$ equipped with a nondegenerate symmetric form, whose cohomology ring coincides with that of $OG(n,2n)$. Choose any Kähler metric on $OG(n-1,2n-1)$. By abuse of notation we use Φ_λ to denote the harmonic Schubert form corresponding to the strict partition $\lambda \subset \rho_{n-1}$; in this way the statement of the next theorem will include this example.

Let ω denote the fundamental form of a hermitian metric h on X which is given in any local holomorphic coordinate chart (z_i) as

$$\omega = \frac{i}{2}\sum_{i,j} h\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right) dz_i \wedge d\bar{z}_j.$$

In each example we normalize the hermitian metric so that its fundamental form coincides with the unique Schubert form of type $(1,1)$. We can now state our main result computing the Hodge star operator in examples (i)–(iv). The two exceptional cases will be discussed in Section 3.

Theorem 1. *The action of the Hodge star operator on the Schubert forms in examples* (i)–(iii) *is as follows*:

$$*\Omega_\lambda = \frac{h_\lambda}{h_{\lambda'}}\Omega_{\lambda'}, \quad *\Phi_\lambda = \frac{g_\lambda}{g_{\lambda'}}\Phi_{\lambda'}, \quad *\Psi_\lambda = 2^{\alpha(\lambda')-\alpha(\lambda)}\frac{g_\lambda}{g_{\lambda'}}\Psi_{\lambda'}.$$

In example (iv) *if* $n=2k-1$ *is odd we have*

$$*\omega^i = \frac{2\cdot i!}{(n-i)!}\omega^{k-1-i}e, \quad 0 \leqslant i \leqslant k-1$$

and if $n=2k$ *is even then*

$$*\omega^i = \frac{2\cdot i!}{(n-i)!}\omega^{k-i}e_0, \quad 0 \leqslant i \leqslant k-1 \quad \textit{and} \quad *e_j = e_{j+k}$$

while the remaining terms are determined by the relation $**=1$.

2. Schubert forms on homogeneous spaces

We recall some fundamental facts from Kostant's seminal papers [11,12]. We derive from Kostant's results that a Schubert form on an irreducible compact hermitian symmetric space is mapped by the Hodge star operator to a multiple of the dual Schubert form. We consider the following situation: $\mathfrak{g}$ is a complex semi-simple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ which contains a fixed Borel subalgebra $\mathfrak{b}$, $\mathfrak{n}$ the maximal nilpotent ideal of $\mathfrak{p}$, and $\mathfrak{k}$ a fixed compact real form of $\mathfrak{g}$. The choice of $\mathfrak{k}$ determines a Cartan involution $x \mapsto x^\theta$ on $\mathfrak{g}$ defined as $(u+iv)^\theta = u-iv$ for $u,v\in\mathfrak{k}$.For any subspace $\mathfrak{s}$ of $\mathfrak{g}$, we set $\mathfrak{s}^\theta = \{x^\theta \mid x\in\mathfrak{s}\}$. Let $\mathfrak{g}_1 = \mathfrak{p}\cap\mathfrak{p}^\theta \subset \mathfrak{g}$. We have $\mathfrak{p}=\mathfrak{g}_1+\mathfrak{n}$ and $\mathfrak{g}=\mathfrak{n}+\mathfrak{g}_1+\mathfrak{n}^\theta$. Let $\mathfrak{r}=\mathfrak{n}+\mathfrak{n}^\theta$ and $\mathfrak{r}_{\mathbb{R}}=\mathfrak{r}\cap\mathfrak{k}\subseteq\mathfrak{g}$. The subspace $\mathfrak{r}_{\mathbb{R}}$ defines a real structure on $\mathfrak{r}$.

Let G be a connected and simply connected complex Lie group with Lie algebra $\mathfrak{g}$, P the closed connected subgroup of G with Lie algebra $\mathfrak{p}$, K the maximal compact subgroup of G corresponding to $\mathfrak{k}$, and V the closed subgroup $K\cap P$ of K. The coset space $X=G/P$ is a compact complex algebraic homogeneous space of positive Euler characteristic and every such space is of this form. The inclusion of K into G induces a diffeomorphism from K/V to G/P. We assume in the following that $X=K/V$ is equipped with a K-invariant hermitian metric. This metric induces the Hodge inner product on the space $A^*(X)$ of smooth complex valued differential forms on X. Let $\Delta = d^*d+dd^*$ be the associated Laplace operator on $A^*(X)$. We equip the space of harmonic forms $\mathscr{H}^*(X)=\ker(\Delta)$ with the induced hermitian metric from $A^*(X)$. The harmonic forms are contained in the subspace $A^*(X)^K$ of K-invariant forms. The natural inclusion from $A^*(X)^K$ to $A^*(X)$ induces a quasi-isomorphism of complexes

$$(A^*(X)^K,d)\to(A^*(X),d).$$

The cohomology of these complexes can be calculated as follows. The projection from G to X induces on (real) tangent spaces a surjection T from $\bigwedge_{\mathbb{R}}\mathfrak{g}$ to $\bigwedge_{\mathbb{R}}T_eX$. The restriction of T defines an isomorphism between $\bigwedge_{\mathbb{R}}\mathfrak{r}_{\mathbb{R}}$ and $\bigwedge_{\mathbb{R}}T_eX$ [12, Lemma 6.7]. Let $T^*_{e,\mathbb{C}}X = T^*_eX\otimes_{\mathbb{R}}\mathbb{C}$ be the space of all complex covectors at the origin of X. There is a unique isomorphism

$$A:\wedge\mathfrak{r}\to\wedge T^*_{e,\mathbb{C}}X \tag{2}$$

defined so that $\langle Au, Tv\rangle = (u,v)_{\mathfrak{g}}$ holds for all $u \in \wedge\mathfrak{r}$ and $v \in \bigwedge_{\mathbb{R}} \mathfrak{r}_{\mathbb{R}}$. Here $(.,.)_{\mathfrak{g}}$ denotes the bilinear form on $\wedge\mathfrak{g}$ induced by the Killing form of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ acts on $\wedge\mathfrak{g}$ by the adjoint representation. The subspace $\wedge\mathfrak{r}$ is stable under the restriction of this representation to $\mathfrak{g}_1$, i.e. $\wedge\mathfrak{r}$ has the structure of a $\mathfrak{g}_1$-module. We denote the subspace $(\wedge\mathfrak{r})^{\mathfrak{g}_1}$ of $\mathfrak{g}_1$-invariant elements by C. Let $d \in \mathrm{End}(\wedge\mathfrak{g})$ be the coboundary operator on $\mathfrak{g}$, that is, the negative adjoint of the Chevalley–Eilenberg boundary operator on $\wedge\mathfrak{g}$ with respect to the Killing form on $\mathfrak{g}$. The coboundary operator d induces a differential d on C. It is well known [12, 6.9] that one obtains an isomorphism of differential graded algebras

$$(A^*(X)^K, d) \tilde{\to} (C, d) \tag{3}$$

by mapping an invariant differential form ω to its restriction $\omega|_e \in (\wedge T^*_{e,\mathbb{C}}X)^{\mathfrak{g}_1} = C$. We obtain a canonical isomorphism of graded rings between $H^*(X,\mathbb{C})$ and $H(C,d)$.

The space $\mathfrak{r}$ is not a Lie subalgebra of $\mathfrak{g}$. However the subalgebras $\mathfrak{n}$ and $\mathfrak{n}^\theta$ in the Lie algebra $\mathfrak{g}$ equip $\mathfrak{r}$ with a Lie algebra structure such that $[\mathfrak{n},\mathfrak{n}^\theta]=0$. Let $\partial \in \mathrm{End}(\wedge\mathfrak{r})$ be the Chevalley–Eilenberg boundary operator for the Lie algebra $\mathfrak{r}$. Let $b \in \mathrm{End}(\wedge\mathfrak{r})$ be the corresponding coboundary operator, that is, the negative adjoint of ∂ with respect to the restriction of $(.,.)_{\mathfrak{g}}$ to $\wedge\mathfrak{r}$. The operators b and ∂ induce operators on C. We consider the Laplacians

$$S = d\partial + \partial d \in \mathrm{End}(C),$$

$$L = b\partial + \partial b \in \mathrm{End}(C).$$

In general, the operator ∂ is not adjoint to d with respect to any hermitian metric on X [6, 3.20]. However Kostant shows in [12, Section 4] that d and ∂ are disjoint, i.e. $d\partial(x)=0$ implies $\partial(x)=0$ and $\partial d(y)=0$ implies $d(y)=0$ for all $x,y \in C$. This implies that the kernel of S computes the cohomologies $H(C,d)$ and $H(C,\partial)$. We obtain canonical isomorphisms

$$\psi_{\Delta,S} : \ker(S) \tilde{\to} H(C,d) \tilde{\to} \ker(\Delta)$$

and

$$\psi_{S,L} : \ker(L) \tilde{\to} H(C,\partial) \tilde{\to} \ker(S).$$

Using the involution determined by the compact real form $\mathfrak{k}$ of $\mathfrak{g}$, we can define a positive definite hermitian inner product on $\wedge\mathfrak{g}$ by

$$\{u,v\} = (-1)^{\deg(u)}(u, v^\theta)_{\mathfrak{g}}$$

for all $u,v \in \wedge\mathfrak{g}$ [11, 3.3]. We equip the subspaces $\wedge\mathfrak{r}$ and C of $\wedge\mathfrak{g}$ with the induced hermitian inner product. We are going to describe an orthogonal basis of the subspace $\ker(L)$ of C. Therefore we consider the representation of $\mathfrak{g}_1$ on the Lie algebra homology $H_*(\mathfrak{n})$ induced by the adjoint action of $\mathfrak{g}$ on $\wedge\mathfrak{g}$. This representation has the following description. Let $\mathfrak{h}$ denote the Cartan subalgebra $\mathfrak{b}\cap\mathfrak{b}^\theta$ of $\mathfrak{g}$, $R=R(\mathfrak{g},\mathfrak{h})$ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The choice of the Borel subalgebra $\mathfrak{b}$ determines subsets R_+ and R_- of R of positive and negative roots, respectively. The maximal nilpotent ideal $\mathfrak{n}$ is an $\mathfrak{h}$-module under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We denote by $R(\mathfrak{n})$ the set of all roots whose root spaces lie in $\mathfrak{n}$. Let W be the Weyl group of $\mathfrak{g}$. For every $\sigma \in W$, we have a subset $\Phi_\sigma = (\sigma R_-)\cap R_+$ of the set of roots R. Corresponding to the parabolic subalgebra $\mathfrak{p}$, we define the set

$$W^1 = \{\sigma \in W \mid \Phi_\sigma \subset R(\mathfrak{n})\}. \tag{4}$$

According to [11, Corollary 8.1] the $\mathfrak{g}_1$-module $H_*(\mathfrak{n})$ may be decomposed as

$$H_*(\mathfrak{n}) = \bigoplus_{\sigma\in W^1} M_\sigma,$$

where each M_σ is an irreducible $\mathfrak{g}_1$-module such that M_σ is not isomorphic to M_τ for $\sigma \neq \tau$. The isomorphism $\wedge\mathfrak{r} = \wedge\mathfrak{n} \otimes \wedge\mathfrak{n}^\theta$ induces an isomorphism [12, Proposition 3.10]

$$H(C,\partial) = (H_*(\mathfrak{n}) \otimes H_*(\mathfrak{n}^\theta))^{\mathfrak{g}_1}.$$

The Killing form of $\mathfrak{g}$ puts $H_*(\mathfrak{n})$ and $H_*(\mathfrak{n}^\theta)$ in duality, so that $H_*(\mathfrak{n})$ is the representation dual to $H_*(\mathfrak{n}^\theta)$. Using Schur's lemma, we get

$$\begin{aligned}(H_*(\mathfrak{n}) \otimes H_*(\mathfrak{n}^\theta))^{\mathfrak{g}_1} &= \mathrm{Hom}_{\mathfrak{g}_1}(H_*(\mathfrak{n}), H_*(\mathfrak{n}))\\ &= \bigoplus_{\sigma\in W^1} \mathrm{Hom}_{\mathfrak{g}_1}(M_\sigma, M_\sigma)\\ &= \bigoplus_{\sigma\in W^1} C_\sigma,\end{aligned}$$

where each C_σ is a one-dimensional space. Let h_σ be the preimage in $\ker(L)$ of a generator of C_σ under the isomorphism

$$\ker(L) = H(C,\partial) = (H_*(\mathfrak{n}) \otimes H_*(\mathfrak{n}^\theta))^{\mathfrak{g}_1} = \bigoplus_{\sigma\in W^1} C_\sigma.$$

The elements h_σ form an orthogonal basis of the subspace $\ker(L)$ of C [12, Theorem 5.4]. There is a canonical way to normalize the choice of h_σ [12, Proposition 5.5] which is not needed in the following. The class of a suitable multiple ω_σ of the image of h_σ under the isomorphism

$$\ker(L) \overset{\psi_{S,L}}{\tilde{\to}} \ker(S) \overset{\psi_{\Delta,S}}{\tilde{\to}} \ker(\Delta) = \mathscr{H}^*(X). \tag{5}$$

in $H^*(X,\mathbb{C})$ is the Poincaré dual of the Schubert cell in $X = G/P$ corresponding to σ [12, Theorem 6.15]. We call ω_σ the *Schubert form* corresponding to σ. It follows from Schubert calculus that for each $\sigma \in W^1$ there is a unique $\sigma' \in W^1$ such that

$$\int_X \omega_\tau \wedge \omega_\sigma \neq 0 \Leftrightarrow \tau = \sigma' \tag{6}$$

holds for all $\tau \in W^1$. We call $\omega_{\sigma'}$ the Poincaré dual form to ω_σ (an exact expression for σ' is given in [4, Corollary 2.6]). Recall that the Hodge-$*$-operator on $A^*(X)$ is determined by

$$\alpha \wedge *\bar{\beta} = \{\alpha, \beta\}\mu_X,$$

where $\mu_X = (n!)^{-1}\omega^n$ is the normalized top exterior power of the fundamental form ω of the metric. Let us assume that the map (5) is an isometry. Under this assumption, the Schubert forms ω_σ, $\sigma \in W^1$, form an orthogonal basis of $\mathscr{H}^*(X)$. Furthermore the Schubert forms are real and have the property (6). It follows that the Hodge $*$-operator maps a Schubert form ω_σ to a multiple of $\omega_{\sigma'}$.

Let us finally assume that X is an irreducible compact hermitian symmetric space. We equip X with its standard homogeneous hermitian metric. This is the unique K-invariant hermitian metric on X for which (2) becomes an isometry. For compact hermitian symmetric spaces, this metric is Kähler and the space of harmonic forms $\mathscr{H}^*(X)$ coincides with the space $A^*(X)^K$ of K-invariant forms. Furthermore, the Lie algebra $\mathfrak{n}$ is commutative and the differentials d, ∂ and b vanish on C. We see in particular that $\psi_{S,L}$ is the identity and $\psi_{\Lambda,S}$ coincides with (3). It follows that (5) becomes an isometry as (3) is an isometry. Thus we have established:

Proposition 1. *Let X be an irreducible hermitian symmetric space of compact type. Then the Hodge star operator maps a Schubert form ω_σ to a non-zero multiple of the Poincaré dual form $\omega_{\sigma'}$.*

The example in Section 5 will show that the analogue of Proposition 1 fails to hold for arbitrary homogeneous spaces of the type G/P considered above.

3. Proof of the main theorem

Recall that for any hermitian compact manifold X the Lefschetz operator $L : A^*(X) \to A^{*+2}(X)$ on the space of smooth complex valued differential forms is given by $L(\eta) = \omega \wedge \eta$, where ω is the fundamental form of the metric. When X is a hermitian symmetric space ω is normalized to coincide with the unique Schubert form of type $(1,1)$.

Let $\Lambda : A^{*+2}(X) \to A^*(X)$ be the adjoint of L with respect to the Hodge inner product on $A^*(X)$. Recall that a differential form is *primitive* if it lies in the kernel of Λ. If $\dim_{\mathbb{C}} X = d$ and $\phi \in A^{p,p}(X)$ is a primitive form, then

$$*L^r\phi = (-1)^p \frac{r!}{(d-2p-r)!} L^{d-2p-r}\phi. \tag{7}$$

This follows from a more general theorem due to Weil [23, Chapter 1, Theorem 2]. Applying (7) when $\phi = 1$ (and $p = 0$) gives

$$*\omega^r = \frac{r!}{(d-r)!}\omega^{d-r}. \tag{8}$$

The rest of the argument is a case by case analysis:

(i) $X = G(m,n)$. In this case $\omega = \Omega_1$ and we have

$$\Omega_1^r = \sum_{|\lambda|=r} \frac{r!}{h_\lambda}\Omega_\lambda \tag{9}$$

(see for instance [15, Example I.4.3]). Using this in (8) gives

$$\sum_{|\lambda|=r} \frac{*\Omega_\lambda}{h_\lambda} = \sum_{|\mu|=mn-r} \frac{\Omega_\mu}{h_\mu}. \tag{10}$$

It follows from Proposition 1 that $*\Omega_\lambda = r_\lambda \Omega_{\lambda'}$ for some real number r_λ. Since the Ω_μ are linearly independent, (10) implies that $r_\lambda = h_\lambda / h_{\lambda'}$, as required.

(ii) $X = OG(n,2n)$. The analogue of equation (9) here is

$$\Phi_1^r = \sum_{|\lambda|=r} \frac{r!}{g_\lambda}\Phi_\lambda.$$

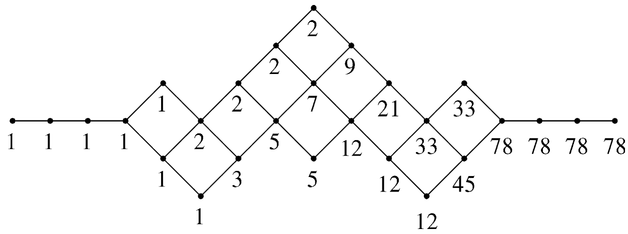

Fig. 4. The poset of Schubert forms for $E_6/(SO(10)\cdot SO(2))$.

This follows from the Pieri formula for X (see [8,9,16, Section 6].) The rest of the argument is the same as in case (i).

(iii) $X = LG(n,2n)$. The Pieri rule of [9] gives

$$\Psi_1^r = \sum_{|\lambda|=r} 2^{\alpha(\lambda)} \frac{r!}{g_\lambda} \Psi_\lambda$$

and we obtain the result as in the previous two cases.

(iv) $X = Q(n)$. If $n=2k-1$ (resp. $n=2k$) then for $i \leqslant k-1$ we have $\omega^{n-i} = 2\omega^{k-1-i}e$ (resp. $\omega^{n-i} = 2\omega^{k-i}e_0 = 2\omega^{k-i}e_1$) and the result follows immediately from (8). The formulas for ω^{n-i} are easily deduced from the formulas for ω^k in Section 1. If $n = 2k$ then (8) gives $*\,\omega^k = \omega^k$ and hence

$$*e_0 + *e_1 = e_0 + e_1.$$

But Proposition 1 implies that $*e_j$ is a multiple of e_{j+k} for $j = 0, 1$. Since e_0, e_1 freely generate $\mathscr{H}^{2k}(X)$, we must have $*e_j = e_{j+k}$. $\square$

The above argument applies to the exceptional spaces (v), (vi) of Section 1 as well. In general, the Schubert forms are parametrized by the Bruhat partially ordered set W^1, defined in (4). Each Schubert form α corresponds to a node in the Bruhat poset; let $N(\alpha)$ denote the number of paths connecting 1 to α in W^1. Fig. 4 shows the poset W^1 and the numbers $N(\alpha)$ in case (v). If α is a $2|\alpha|$-form and α_1 the unique Schubert $(1,1)$-form then

$$\alpha_1^r = \sum_{|\alpha|=r} N(\alpha)\alpha$$

hence (8) gives

$$*\alpha = \frac{|\alpha|!N(\alpha')}{|\alpha'|!N(\alpha)}\alpha' \tag{11}$$

where α' denotes the Poincaré dual of α. Using (11) one can compute the action of $*$ in examples (v) and (vi).

Remarks on normalization. (1) (Forms) In example (i) if we renormalize by setting $\tilde{\Omega}_\lambda = \Omega_\lambda/h_\lambda$ then Theorem 1 gives $*\tilde{\Omega}_\lambda = \tilde{\Omega}_{\lambda'}$. However in this case $\tilde{\Omega}_{\lambda'}$ is not the Poincaré dual of $\tilde{\Omega}_\lambda$. A similar comment applies to the other spaces considered.

(2) (Metrics) Set $\omega' = \rho\,\omega$ for some $\rho > 0$ and let $*$ (respectively, $*'$) denote the Hodge star operator associated with ω (respectively, ω'). Then $*' = \rho^{d-k} *$ on the vector space $A^k(X)$ of differential k-forms on X.

4. Formulas for the adjoint of the Lefschetz operator

In this section we provide an explicit computation of the adjoint Λ of the Lefschetz operator in the Grassmannian examples of Section 1. This calculation was done in a different way by Proctor [17–19] in the case of minuscule flag manifolds. It follows from the definition of Λ that

$$\Lambda = *L* \tag{12}$$

as all non-zero harmonic forms occur in even degrees. Consequently one can use our calculation for $*$ to find Λ.

The Schubert forms in all our examples form a partially ordered set, with the order induced from the Bruhat order on the underlying Weil group (in combinatorial language they form an *irreducible Bruhat poset*, see [18, Section 2]). The Lefschetz operator L (respectively, its adjoint Λ) is an order raising operator (resp. order lowering operator) on the space of harmonic forms $\mathscr{H}^*(X)$. We will work out each of the three cases separately:

(i) $X = G(m,n)$. The action of L on the Schubert forms is given by the Pieri rule:

$$L(\Omega_\lambda) = \sum_\mu \Omega_\mu$$

the sum over all $\mu \supset \lambda$ with $|\mu| = |\lambda| + 1$ (as usual, the inclusion relation on partitions is defined by the containment of diagrams.) Now Eq. (12) and Theorem 1 give

$$\Lambda(\Omega_\lambda) = \sum_\mu e_{\lambda\mu}(m,n)\Omega_\mu$$

the sum over all μ with $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$, with $e_{\lambda\mu}(m,n) = h_\lambda h_{\mu'}/h_{\lambda'} h_\mu$.

Proposition 2. *For all $\mu \subset \lambda$ with $|\mu| = |\lambda| - 1$, we have*

$$e_{\lambda\mu}(m,n) = (m - i + \lambda_i)(n + i - \lambda_i),$$

where i is the unique index such that $\mu_i = \lambda_i - 1$.

Proof. For any partition λ with $\ell(\lambda) \leqslant m$, the *$\beta$-sequence* β^λ is defined as the m-tuple

$$\beta^\lambda = (\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m + m - m).$$

It is shown in [15, Example I.1.1] that

$$h_\lambda = \frac{\prod_j \beta_j^\lambda!}{\prod_{j<k}(\beta_j^\lambda - \beta_k^\lambda)}.$$

Note that since

$$\lambda' = (n - \lambda_m, n - \lambda_{m-1}, \ldots, n - \lambda_1)$$

we have

$$\beta^{\lambda'} = (m+n-1-\lambda_m, m+n-2-\lambda_{m-1}, \ldots, m+n-m-\lambda_1).$$

We now claim that

$$\prod_{j<k}(\beta_j^{\lambda} - \beta_k^{\lambda}) = \prod_{j<k}(\beta_j^{\lambda'} - \beta_k^{\lambda'}).$$

Indeed, the absolute value of these products is invariant under translation $t_p(x) := x + p$ and inversion $i(x) := -x$ of the entire β-sequence, and

$$t_{m+n-1}(i(\beta^{\lambda})) = \beta^{\lambda'}.$$

It follows that

$$e_{\lambda\mu}(m,n) = \frac{h_{\lambda}h_{\mu'}}{h_{\lambda'}h_{\mu}} = \frac{\prod_j \beta_j^{\lambda}!\prod_j \beta_j^{\mu'}!}{\prod_j \beta_j^{\lambda'}!\prod_j \beta_j^{\mu}!}.$$

Finally, it is easy to check that

$$\frac{\prod_j \beta_j^{\lambda}!}{\prod_j \beta_j^{\mu}!} = m - i + \lambda_i \quad \text{and} \quad \frac{\prod_j \beta_j^{\mu'}!}{\prod_j \beta_j^{\lambda'}!} = n + i - \lambda_i$$

as only one β-number (β_i) changes (by one unit) when we pass from λ to μ. □

(ii) $X = OG(n,2n)$. In this case the Lefschetz operator satisfies

$$L(\Phi_{\lambda}) = \sum_{\mu} \Phi_{\mu}$$

the sum over all strict partitions μ with $\lambda \subset \mu \subset \rho_{n-1}$ and $|\mu| = |\lambda| + 1$. Theorem 1 and Eq. (12) are now used to show that

$$\Lambda(\Phi_{\lambda}) = \sum_{\mu} f_{\lambda\mu}(n)\Phi_{\mu}$$

the sum over μ with $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$, where $f_{\lambda\mu}(n) = g_{\lambda}g_{\mu'}/g_{\lambda'}g_{\mu}$.

Proposition 3. *For all strict $\mu \subset \lambda$ with $|\mu| = |\lambda| - 1$, we have*

$$f_{\lambda\mu}(n) = \begin{cases} n(n-1)/2 & \textit{if } k = 1, \\ n(n-1) - k(k-1) & \textit{otherwise}, \end{cases}$$

where k is the unique part of λ which is not a part of μ.

Proof. The numbers g_{λ} satisfy

$$g_{\lambda} = \prod_i \lambda_i! \cdot \frac{\prod_{i<j}(\lambda_i + \lambda_j)}{\prod_{i<j}(\lambda_i - \lambda_j)}. \tag{13}$$

This formula is due to Schur [20]; see also [15, Example III.8.12]. Now assume that $\lambda\setminus\mu=\{k\}$ and suppose that $k>1$. Let us compute the contribution of the three terms in (13) to the quotient $g_\lambda g_{\mu'}/g_{\lambda'}g_\mu$: the terms $\prod\lambda_i!$ contribute

$$\frac{\prod\lambda_i!\,\prod\mu_i'!}{\prod\mu_i!\,\prod\lambda_i'!}=\frac{k!k!}{(k-1)!(k-1)!}=k^2. \tag{14}$$

The contribution of the terms $\prod_{i<j}(\lambda_i+\lambda_j)$ is given by

$$\frac{\prod_{j\notin\{k,k-1\}}(k+j)}{\prod_{j\notin\{k,k-1\}}(k-1+j)}=\frac{(k-1)(k+n-1)}{k^2}. \tag{15}$$

The contribution of the terms $\prod_{i<j}(\lambda_i-\lambda_j)$ is given by

$$\frac{\prod_{j\neq k}|k-1-j|}{\prod_{j\neq k}|k-j|}=\frac{n-k}{k-1}. \tag{16}$$

Multiplying (14), (15) and (16) together gives

$$f_{\lambda\mu}(n)=\frac{g_\lambda g_{\mu'}}{g_{\lambda'}g_\mu}=(n-k)(k+n-1).$$

The case $k=1$ is handled similarly. □

(iii) $X=LG(n,2n)$. The Lefschetz operator on $\mathscr{H}^*(X)$ is given by

$$L(\Psi_\lambda)=2\sum_\mu\Psi_\mu+\Psi_{\lambda^+},$$

where the sum is over all (strict) μ obtained from λ by adding a box in a non-empty row and $\lambda^+=(\lambda_1,\dots,\lambda_{\ell(\lambda)},1)$ (this follows from the more general Pieri rule given by Hiller and Boe [9]). The computation of the adjoint operator Λ here is similar to the previous two cases, so we simply state the answer

$$\Lambda(\Psi_\lambda)=\frac{1}{2}\sum_\mu f_{\lambda\mu}(n+1)\Psi_\mu+f_{\lambda\lambda^-}(n+1)\Psi_{\lambda^-},$$

where the sum is over all μ with $\ell(\mu)=\ell(\lambda)$ obtained from λ by subtracting a box. The partition λ^- is defined to be empty if 1 is not a part of λ, and otherwise $\lambda^-=\lambda\setminus 1$. Note that this calculation was not given by Proctor, as X is not a minuscule flag manifold.

We omit the computation of Λ for the remaining hermitian symmetric spaces, which may be done in a similar fashion. The resulting coefficients can be found in [19].

For any Kähler manifold X define the endomorphism $B:\mathscr{H}^*(X)\to\mathscr{H}^*(X)$ by

$$B=\sum_{i=0}^{2d}(d-i)pr_i,$$

where pr_i is the projection onto the ith homogeneous summand of $\mathscr{H}^*(X)$. It is well known that the operators L, Λ and B satisfy the commutator relations

$$[B,L] = -2L, \quad [B,\Lambda] = 2\Lambda, \quad [\Lambda,L] = B$$

and hence we get a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathscr{H}^*(X)$. It follows that Λ is completely determined by L and B; see for instance [17, Proposition 2] for a proof. Weil's formula (7) and the Lefschetz decomposition theorem now imply that $* : \mathscr{H}^*(X) \to \mathscr{H}^*(X)$ is completely determined by the group $\mathscr{H}^*(X)$ together with the action of the Lefschetz operator. This allows us to include Kähler manifolds like example (ii′) in our results.

5. An example

We calculate the action of the Hodge star operator for the complete flag manifold

$$F = F_{1,2,3}(\mathbb{C}^3) = SU(3)/S(U(1)^3)$$

which parametrizes complete flags in a three dimensional complex vector space. We will see that the analogue of Proposition 1 fails when F is equipped with any $SU(3)$-invariant metric.

There is a universal vector bundle E over F together with a tautological filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq E_3 = E$$

by subbundles such that each quotient $L_i = E_i/E_{i-1}$ is a line bundle on F. We consider the natural $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[x_1,x_2,x_3]$ to $H^*(F,\mathbb{C})$ which maps x_i to $y_i = -c_1(L_i) \in H^2(F,\mathbb{C})$. It is well known that this map is surjective and that its kernel is generated by the elementary symmetric polynomials $x_1+x_2+x_3$, $x_1x_2+x_1x_3+x_2x_3$, and $x_1x_2x_3$. This yields relations $y_3 = -y_1 - y_2$, $y_2^2 = -y_1^2 - y_1y_2$, and $y_1^3 = 0$ in $H^*(F,\mathbb{C})$. The cycle classes of the Schubert varieties in F define an $\mathbb{C}$-basis of $H^*(F,\mathbb{C})$. This basis is given by the classes 1, y_1, y_1+y_2, y_1^2, y_1y_2, and $y_1^2y_2$ (these are the *Schubert polynomials* for S_3; see [14]).

Equip F with any $SU(3)$-invariant hermitian metric and denote the fundamental form of this metric by ω. We will show that the Hodge-$*$-operator associated with ω satisfies

$$*y_1 = \lambda y_1^2 + \mu(y_1y_2)$$

for some $\lambda \neq 0$. Observe that this is in contrast with our result in the hermitian symmetric space case as y_1y_2 is Poincaré dual to y_1.

We will work with the differential (1,1)-forms Ω_{ij}, $1 \leqslant i < j \leqslant 3$ on F constructed in [22, Section 5]. The Ω_{ij} are a basis of the three-dimensional space of $SU(3)$-invariant forms in $A^2(F)$; note that there are no $SU(3)$-invariant 1-forms on F. The subspace of harmonic forms $\mathscr{H}^2(F)$ in $A^2(F)$ has dimension two. We have

$$\omega = \alpha\,\Omega_{12} + \beta\Omega_{13} + \gamma\Omega_{23}$$

for some positive real numbers α,β,γ; the metric is Kähler if and only if $\beta = \alpha + \gamma$ (see for instance [2, Section 4]).

The harmonic representatives h_i of the classes y_i do not depend on the choice of invariant metric, and are given as [22, Corollary 3]

$$h_1 = \Omega_{12} + \Omega_{13}, \quad h_2 = -\Omega_{12} + \Omega_{23}, \quad h_3 = -\Omega_{13} - \Omega_{23}.$$

One checks easily that $z = \alpha\Omega_{12} - \beta\Omega_{13}$ and $z' = \gamma\Omega_{23} - \alpha\Omega_{12}$ are ω-primitive, i.e. satisfy $\omega^2 \wedge z = 0$ and $\omega^2 \wedge z' = 0$ in $A^*(F)$. Using formulas (7), (8), and the equalities

$$3\alpha\Omega_{12} = \omega + z - z', \quad 3\beta\Omega_{13} = \omega - 2z - z', \quad 3\gamma\Omega_{23} = \omega + z + 2z'$$

one calculates

$$*\alpha\Omega_{12} = \beta\gamma\Omega_{13} \wedge \Omega_{23}, \quad *\beta\Omega_{13} = \alpha\gamma\Omega_{12} \wedge \Omega_{23}, \quad *\gamma\Omega_{23} = \alpha\beta\Omega_{12} \wedge \Omega_{13}$$

and

$$*h_1 = \frac{\alpha\gamma}{\beta}\Omega_{12} \wedge \Omega_{23} + \frac{\beta\gamma}{\alpha}\Omega_{13} \wedge \Omega_{23}.$$

We conclude that

$$\lambda = \int_F (*h_1) \wedge (h_1 + h_2) = \frac{\alpha\gamma}{\beta}\int_F \Omega_{12} \wedge \Omega_{13} \wedge \Omega_{23} \neq 0.$$

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